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Mean-Median Compromise Method: A Novel Deepest Voting Function Balancing Range Voting and Majority Judgment

Ruffin-Benoît M. Ngoie [1,2,*], Selain K. Kasereka [2,3], Jean-Aimé B. Sakulu [1] and Kyandoghere Kyamakya [4]

1 Department of Mathematics, Institut Supérieur Pédagogique de Mbanza-Ngungu, Mbanza-Ngungu B.P. 127, Democratic Republic of the Congo; jaimesakulu@isp-mbng.ac.cd
2 Artificial Intelligence, Big Data and Modeling Simulation Research Center (ABIL), Kinshasa B.P. 190, Democratic Republic of the Congo; selain.kasereka@unikin.ac.cd
3 Mathematics, Statistics and Computer Science Department, University of Kinshasa, Kinshasa B.P. 190, Democratic Republic of the Congo
4 Institute of Smart Systems Technologies, University of Klagenfurt, 9020 Klagenfurt, Austria; kyandoghere.kyamakya@aau.at
* Correspondence: rbngoie@isp-mbng.ac.cd; Tel.: +243-897-111-489

Abstract: A logical presentation of the Mean-Median Compromise Method (MMCM) is provided in this paper. The objective is to show that the method is a generalization of majority judgment, where each tie-break step is L^p deepest voting. Therefore, in its tie-breaking procedures, the proposed method returns scores that range from the median to the mean. Among the established characteristics that it satisfies are universality, neutrality, independence of irrelevant alternatives, unanimity, and monotonicity. Additionally covered are robustness, reaching consensus, controlling extremes, responding to single-peakedness, and the impact of outliers. Through computer simulations, it is shown that the MMCM score does not vary by more than 12% even for up to 50% of strategic voters, ensuring the method's robustness. The 1976 Paris wine taste along with the French presidential poll organized by OpinionWay in 2012 were well-known and highly regarded situations in the area of social choice to which the MMCM was used. The outcomes of MMCM have shown remarkable consistency. On the basis of the democratic standards that are most frequently discussed in the literature, other comparisons were performed. With 19 of the 25 criteria satisfied, the MMCM is in the top ranking. Supporting theorems have shown that MMCM does not necessarily require an absolute majority to pass an opinion for which a minority expresses a strong preference while the majority is only marginally opposed.

Keywords: decision-making; democratic processes; electoral systems; mean-median compromise method; preference aggregation; voting mechanisms

MSC: 00A06; 91A80; 91B12; 91B14; 91-02; 91A35

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1. Introduction

In a community that claims to be democratic, the allocation of social or political responsibilities is based on the voting procedure. The latter is the act by which an individual in the community expresses their preference over the proposed candidates. A mapping of voter preferences for a subset of candidates known as "winners" is what constitutes an election.

Since the mid-1900s, it has been known, thanks to the work of Arrow [1], that no social choice rule, regardless of complexity, can satisfy a relatively narrow set of democratic conditions. This discovery, sometimes known as "Arrow's Impossibility Theorem" or the "General Possibility Theorem", applies specifically to ordinal preferences, where voters rank options without assigning explicit scores. In contrast, methods like majority judgment (MJ), range voting (RV), and approval voting (AV) use cardinal preferences, where voters provide ratings or approval levels. The axiomatic line of research in social choice theory has thus broadened to explore these alternative ballot types, seeking solutions that balance fairness and representativeness. In [2,3], the authors propose axiomatic approaches to voting methods incorporating cardinal preferences that aim to provide a more nuanced understanding of voter preferences by allowing voters to express the intensity of their preferences, rather than just their order.

In the early 21st century, refs. [3–8] proposed majority judgment (MJ), a voting function for which preferences are cardinal (evaluations) and whose result for each candidate is the median of the evaluations they received from voters. The winner of the MJ is the candidate who gets the highest median. Sadly, even though the MJ defeats Arrow's impossibility thesis, it still has a number of additional flaws (Felsenthal and Machover [9], Felsenthal [10], Felsenthal and Machover [11]). According to Aubin et al. [12], the primary flaw in MJ is that its tie-breaking rule is not suitable.

In fact, range voting (RV), approval voting (AV), and majority judgment (MJ) are all L^p deepest voting, with $p = 1$ for MJ and $p = 2$ for both AV and RV, according to [12]. The vulnerability of the deepest voting to manipulation (strategic voting) increases with increasing p. Nevertheless, AV and RV are also not flawless systems. In the opinion of Balinski and Laraki [3], Aubin et al. [12], Balinski and Laraki [13], the RV is extremely manipulable. The AV's disadvantages are covered in [7].

This paper introduces the Mean-Median Compromise Method (MMCM) [14–16], which is an extension of the MJ and RV with the proper tie-breaking rule. The voting method is a L^p deepest with a recurrent tie-breaking procedure. A L^p deepest voting with $p \in [1, 2]$ is used to break ties.The idea behind a deepest voting is to use the scatter plot's deepest point to determine the winner by representing each voter's grade on d candidates as a point in $\mathbb{R}^d$. A depth function is maximized to determine the lowest spot. The reader is referred to [12] for further information on L^p deepest voting. The criteria of neutrality, universality, unanimity, monotonicity, and independence of irrelevant alternatives (IIA) are all satisfied by MMCM, as is the case with any depth function. Unfortunately, MMCM has some downsides, including the Condorcet winner, Condorcet loser, reinforcement (except in the scenario when $p = 2$), and the no-show paradoxes.

The MMCM bridges the gap between traditional social choice theory and broader scientific fields such as sensory and consumer science, sensometrics, multivariate data analysis, and statistical analysis of rankings and preference data. By incorporating L^p deepest voting in its tie-breaking steps, it covers a spectrum of scores from the median to the mean. This method demonstrates robustness, achieving consensus while controlling extremes and responding effectively to single-peaked preferences and outliers. Its applications to renowned scenarios such as the 1976 Paris wine tasting and the 2012 French presidential poll organized by OpinionWay showcase its consistency and reliability.

The remainder of this paper is organized as follows: In Section 2, important definitions are examined. A number of both MJ and RV's restrictions are listed in Section 3. In Section 4, MMCM is described in a new way. Afterwards, Sections 5 and 6 analyze and discuss the MMCM. In Section 7, MMCM and most valuable voting functions are compared using 25 democratic criteria. The Paris 1976 wine taste and the 2012 French presidential poll organized by OpinionWay—real-world situations—are subjected to MMCM in Section 8, where the results are compared to those obtained from alternative voting schemes. In Section 9, we discuss MMCM's robustness, sensitivity, and implications for various voting contexts. Final thoughts are included in Section 10.

2. Preliminaries

2.1. Ranking Framework

Let $C = \{c_1, c_2, \ldots, c_m\}$ be the set of m candidates and $V = \{v_1, v_2, \ldots, v_n\}$ be the set of n voters. An ordinal preference or *ranking* is the binary relation $\succ$, defined as follows: $c_i \succ c_j$ if c_i is preferred to c_j. Ordinal preferences constitute the voting paradigm where voters rank (strictly or weakly) candidates from the best to the worst.

2.2. Grading Framework

Suppose that we have m candidates, n voters, and each voter v_i allots to each candidate c_j a grade g_{ij} chosen from a set denoted by Λ. As pointed out by [4], Λ might be discrete, finite, or an interval of real numbers. In this paper, we will consider that Λ is strictly ordered.

A profile is an $m \times n$ matrix $(g_{ij})_{i=1,\ldots,n,j=1,\ldots,m} \in \Lambda^{m\times n}$ of grades g_{ij} assigned by voters to candidates. Hereafter, we denote a profile by ϕ. We can easily notice that $\Lambda = \{0,1\}$ for approval voting and first-past-the-post, while it might be discrete or continuous for RV, MJ and, as will be seen, MMCM.

2.3. Method of Grading

A method of grading can be seen as a function f that assigns to any profile ϕ one final rating for every candidate $f : \Lambda^{m\times n} \to \Gamma^m$ (with $\Lambda \subset \Gamma$). The set Γ of possible grades f returns may differ from Λ [12].

In [4], it is shown that any method of grading satisfies:

- Anonymity: Consider the profile

$$\phi = \begin{bmatrix} g_{11} & g_{21} & \cdots & g_{i1} & \cdots & g_{j1} & \cdots & g_{n1} \\ g_{12} & g_{22} & \cdots & g_{i2} & \cdots & g_{j2} & \cdots & g_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{1m} & g_{2m} & \cdots & g_{im} & \cdots & g_{jm} & \cdots & g_{nm} \end{bmatrix}$$

  and $f(\phi) = (g_1^\star, \ldots, g_m^\star)$. When $\forall i, j = 1, \ldots, n$ swapping the columns of voters v_i and v_j does not alter the final result, the function f is said to be anonymous.
- Neutrality: Suppose two candidates c_x and c_y in competition and n voters $\{v_1, \ldots, v_n\}$. Consider the profile $\phi = \begin{bmatrix} g_{1x} & g_{2x} & \cdots & g_{nx} \\ g_{1y} & g_{2y} & \cdots & g_{ny} \end{bmatrix}$ and $f(\phi) = (g_x^\star, g_y^\star)$ with $g_x^\star > g_y^\star$. The function f is said to be neutral when $\forall i = 1, \ldots, n$ if v_i assigns now g_{iy} to c_x and g_{ix} to c_y, $f(\phi) = (g_y^\star, g_x^\star)$ with $g_y^\star < g_x^\star$.
- Unanimity: Suppose two candidates c_x and c_y in competition and $V = \{v_1, v_2, \ldots, v_n\}$ the set of n voters. Consider the profile

$$\phi = \begin{bmatrix} g_{1x} & g_{2x} & \cdots & g_{nx} \\ g_{1y} & g_{2y} & \cdots & g_{ny} \end{bmatrix}$$

  and f an aggregation function. The function f is said to be unanimous when $\forall v_i \in V$, $g_{ix} > g_{iy} \Rightarrow c_x \succ c_y$.
  If a candidate who receives a greater rating from every voter than the other candidates is declared the winner, the function is considered unanimous.
- Independence of irrelevant alternatives: If the overall community preference between two candidates c_x and c_y is $c_x \succ c_y$, this ranking should remain unchanged by the addition or removal of a candidate c_z, regardless of the grades they receive from voters.
- Monotonicity: Let $V = \{v_1, \ldots, v_n\}$ be the set of voters, and $C = \{c_1, \ldots, c_m\}$ be the set of m candidates. Let $G = \{g_{1j}, \ldots, g_{nj}\}$ be the set of grades assigned by voters v_i to a fixed candidate c_j. Suppose f an aggregation function whose final score for c_j with respect to G is s. If any grade g_{ij} (for $i = 1, \ldots, n$) is replaced by g'_{ij} such that $g'_{ij} \geq g_{ij}$, we denote the new f score as s'. Then, it must hold that $s' \geq s$. A candidate's final score should either rise or remain the same if voters give them a higher rating.

2.4. Deepest Voting

Deepest voting is a method that uses geometric principles to determine the most representative candidate based on voters' grades. Here is how it works:

- Grading matrix: Each voter v_i assigns a grade g_{ij} to each candidate c_j, resulting in a grading matrix G.

- Candidate positioning: Each candidate is represented as a point C_j in an n-dimensional space, where n is the number of voters. The coordinates of C_j are the grades G_{ij} given by each voter v_i.
- Depth function: A depth function $D(C_j)$ is used to determine how central a candidate's position is within the scatter plot of all candidates. One common depth function is the half-space depth, as follows:

$$D(C_j) = \min_u \left| \left\{ \mathbf{x} \in \mathbb{R}^n : u^\top x \leq u^\top C_j \right\} \right|$$

where u is a unit vector, $u^\top$ the transpose of the vector u and $u^\top x$ the dot product (or inner product) of the two vectors $u^\top$ and x.
- Deepest point: The candidate c_j with the highest depth function value $D(C_j)$ is selected as the winner. Mathematically

$$\text{Winner} = \arg\max_j D(C_j)$$

2.5. L^p Depth Function

The L^p depth of a point x in a multivariate distribution is a measure of how central x is within the distribution. Its mathematical definition is as follows: given a point $x \in \mathbb{R}^d$ and a dataset $X = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d$

- Compute the L^p-norm distance between the point x and each data point x_i:

$$d_p(x, x_i) = \left(\sum_{j=1}^{d} |x_j - x_{ij}|^p \right)^{\frac{1}{p}}$$

where x_j and x_{ij} are the j-th components of x and x_i, respectively.
- Depth Function: The L^p depth function $D_p(x)$ is defined as

$$D_p(x) = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{d_p(x, x_i)} \right)^{-1}$$

This depth function gives a measure of centrality for the point x, with higher values indicating a more central position relative to the distribution of data points in the L^p-norm space.

2.6. Majority Judgment

Majority judgment (MJ) evaluates candidates using the median of grades assigned by voters. Let $C = \{c_1, c_2, \ldots, c_m\}$ be the set of candidates and $V = \{v_1, v_2, \ldots, v_n\}$ the set of voters. Each voter v_i assigns a grade g_{ij} to candidate c_j.

2.6.1. Majority Judgment for Small Electorate

When the electorate is small, Balinski and Laraki [4] propose the following procedure for applying the majority judgment:

1. Determine majority grade: For each candidate c_j, determine the majority grade $\tilde{g}_j$. If $G_j = (r_1, r_2, \ldots, r_n)$ (with $r_1 \geq \cdots \geq r_n$) is the sorted list of grades for c_j, then

$$\tilde{g}_j = \begin{cases} r_{(n+1)/2} & \text{if } n \text{ is odd} \\ r_{(n+2)/2} & \text{if } n \text{ is even} \end{cases}$$

2. Tie-breaking: If two candidates receive the same majority grade, the procedure is repeated. In the subsequent iteration, the majority grade is eliminated from each candidate's list of grades.

Example: Consider three candidates c_1, c_2, and c_3, graded by seven voters. The grades are as presented in Table 1.

Table 1. Grades assigned by 7 voters to 3 candidates.

Voter	c_1	c_2	c_3
v_1	4	3	2
v_2	3	4	3
v_3	4	3	4
v_4	4	3	2
v_5	5	5	3
v_6	3	4	4
v_7	4	2	3

Step 1: Determine majority grades

- c_1: sorted grades: [5, 4, 4, 4, 4, 3, 3], majority grade = 4
- c_2: sorted grades: [5, 4, 4, 3, 3, 3, 2], majority grade = 3
- c_3: sorted grades: [4, 4, 3, 3, 3, 2, 2], majority grade = 3

Step 2: Compare majority grades

Rank candidates based on their majority grades, as follows:

$$c_1 \succ c_2 \sim c_3$$

Step 3: Tie-breaking

Candidates c_2 and c_3 have the same majority grade. We eliminate this grade and repeat the procedure for the remaining grades.

- c_2: sorted grades: [5, 4, 4, 3, 3, 2], majority grade = 3
- c_3: sorted grades: [4, 4, 3, 3, 2, 2], majority grade = 3

Candidates c_2 and c_3 are still tied. Again, we drop the majority grade and repeat the procedure.

- c_2: sorted grades: [5, 4, 4, 3, 2], majority grade = 4
- c_3: sorted grades: [4, 4, 3, 2, 2], majority grade = 3

The tie is broken. Therefore, the final ranking is

$$c_1 \succ c_2 \succ c_3$$

2.6.2. Majority Judgment for Large Electorate

In scenarios with multiple judges or voters, such as presidential elections, Balinski and Laraki [2,3] propose a straightforward method for resolving ties. They introduce the majority gauge, represented as the triplet (p_a, α^*, q_a). In this notation, p_a denotes the proportion of evaluations for a candidate that exceed the majority grade, while q_a represents the proportion that fall below it. The majority grade itself is denoted by α.

$$\alpha^* = \begin{cases} \alpha^+ & \text{if } p > q \\ \alpha^- & \text{if } p \leq q \end{cases}$$

α^* is called the "modified majority grade" of the candidate. The ranking of candidates by majority gauge (denoted $\succ_{mg}$) is carried out in the following way: let a and b two candidates with the majority gauges (p_a, α_a^*, q_a) and (p_b, α_b^*, q_b), respectively. Then, $a \succ_{mg} b$ or $(p_a, \alpha_a^*, q_a) \succ_{mg} (p_b, \alpha_b^*, q_b)$ iff $\alpha_a^* > \alpha_b^*$ or ($\alpha_a^* = \alpha_b^* = \alpha^+$ and $p_a > p_b$) or ($\alpha_a^* = \alpha_b^* = \alpha^-$ and $q_a < q_b$).

Example: Around 2000 people were asked to rate the 12 candidates for the Republic's presidency using the following common language: Excellent, Very good, Good, Passable, Poor, and Reject. This was part of the 2007 experiment conducted by Balinski and Laraki [17]. This experiment took place in Orsay, a city near Paris.

The corresponding percentage is bolded to reflect the majority grades of the candidates in the running (see Table 2). Bayrou, Royal, and Sarkozy all have the same grade ("Good"), as can be seen. The majority gauges for Bayrou, Royal, and Sarkozy are, respectively, (44.3%, $Good^+$, 30.6%), (39.4%, $Good^-$, 41.5%), and (38.9%, $Good^-$, 46.9%). Royal $\succ_{mg}$ Sarkozy because 41.5% < 46.9%, and Bayrou $\succ_{mg}$ Royal because $Good^+ > Good^-$.

Table 2. Results of the MJ experiment in 2007 in Orsay.

Candidates	Excellent	Very Good	Good	Passable	Poor	Reject
Besancenot	4.1%	9.9%	16.3%	16.0%	**22.6%**	31.1%
Buffet	2.5%	7.6%	12.5%	20.6%	**26.4%**	30.4%
Schivardi	0.5%	1.0%	3.9%	9.5%	24.9%	**60.4%**
Bayrou	13.6%	30.7%	**25.1%**	14.8%	8.4%	7.4%
Bové	1.5%	6.0%	11.4%	16.0%	**25.7%**	39.5%
Voynet	2.9%	9.3%	17.5%	**23.7%**	26.1%	20.5%
Villiers	2.4%	6.4%	8.7%	11.3%	15.8%	**55.5%**
Royal	16.7%	22.7%	**19.1%**	16.8%	12.2%	12.6%
Nihous	0.3%	1.8%	5.3%	11.0%	26.7%	**55.0%**
Le Pen	3.0%	4.6%	6.2%	6.5%	5.4%	**74.4%**
Laguiller	2.1%	5.3%	10.2%	16.6%	**25.9%**	40.1%
Sarkozy	19.1%	19.8%	**14.3%**	11.5%	7.1%	28.2%

The order determined by the majority gauge is used in Table 3.

Table 3. Ranking of candidates by the majority gauge for the 2007 experiment.

Range	Candidates	p	α^*	q
1	Bayrou	44.3%	Good$^+$	30.6%
2	Royal	39.4%	Good$^-$	41.5%
3	Sarkozy	38.9%	Good$^-$	46.9%
4	Voynet	29.8%	Passable$^-$	46.6%
5	Besancenot	46.3%	Poor$^+$	31.2%
6	Buffet	43.2%	Poor$^+$	30.5%
7	Bové	34.9%	Poor$^-$	39.4%
8	Laguiller	34.2%	Poor$^-$	40.0%
9	Nihous	45.0%	Reject	—
10	Villiers	44.5%	Reject	—
11	Schivardi	39.7%	Reject	—
12	Le Pen	25.7%	Reject	—

2.7. Range Voting

Range voting is a voting method where voters rate each candidate within a specified range (e.g., 0 to 10). The winner is the candidate with the highest average score.

Mathematically, let n be the number of voters and m be the number of candidates. Each voter v_i assigns a score s_{ij} to candidate c_j. The total score T_j for candidate c_j is

$$T_j = \sum_{i=1}^{n} s_{ij}$$

The average score $\bar{s}_j$ for candidate c_j is

$$\bar{s}_j = \frac{T_j}{n}$$

The candidate with the highest average score $\bar{s}_j$ wins

$$\text{Winner} = \max_j \bar{s}_j$$

Example:

Assume there are three voters and three candidates (A, B, and C). Voters give the following scores:

Voter	A	B	C
1	8	5	6
2	7	7	8
3	9	6	7

Calculate the total scores as follows:

$$T_A = 8 + 7 + 9 = 24, T_B = 5 + 7 + 6 = 18 \text{ and } T_C = 6 + 8 + 7 = 21$$

Calculate the average scores as follows:

$$\bar{s}_A = \frac{24}{3} = 8, \bar{s}_B = \frac{18}{3} = 6 \text{ and } \bar{s}_C = \frac{21}{3} = 7$$

Candidate A wins with the highest average score of 8.

3. Some Paradoxical Results for MJ and RV

In this paragraph, we demonstrate a few counterintuitive findings for MJ and RV using examples. These findings offer sufficient evidence that both approaches have advantages and disadvantages and that a compromise solution must be found.

For a survey of the MJ paradoxes, we direct interested readers to [10,18,19]. In the ranking framework, ref. [20] present a voting function akin to MJ. The answers to the reproaches addressed to the MJ are provided by [3,21–23]. However, to date, some responses are still considered unsatisfactory [24,25]. An examination of RV is offered by [3,13].

Example 1. *Consider a, b, and c, three friends who would like to go out to eat together. They have the option of steak or a dish of raw vegetables. Nevertheless, one of them is a vegan and cannot have meat at all. Though they marginally favor beef, non-vegans enjoy all vegetables. Their preferences are as follows:*

	a	*b*	*c*
Steak	9	9	0
Vegetables	8	8	9

The median for veggies is 8, and for beef it is 9. The food the three friends should eat, in the MJ's opinion, is meat. All three companions, though, think the veggie dish is the best. Choosing veggies (with an average score of $\frac{25}{3}$) over meat (with an average score of 6) as advised by the RV (mean-based voting function), would be the best decision. This instance shows that the pathology of majority tyranny is still present in MJ.

Example 2. *Two skaters a and b have their performances judged by five people. The following is how the results table is displayed:*

	j_1	j_2	j_3	j_4	j_5
a	10	5	4	4	3
b	7	7	5	5	1

Since skater a's average score is 5.2 and skater b's is 5, it follows that $a \succ b$. However, three of the five judges—or 60%—think that $b \succ a$. The MJ (median-based) gives $b \succ a$. Due to his excessively high opinion of a, j_1 forced his/her preference on the entire jury. This is inappropriate for a community that aspires to democracy. The best decision would be the one suggested by MJ. One could consider this situation to be a lack of robustness for RV.

Example 3. *In the table below, candidates a and b are evaluated by voters on a scale from 1 to 6, where 6 represents the highest rating and 1 the lowest. The table presents the proportions of voters assigning each specific rating to the candidates. For instance, 21% of voters rated candidate a as 6, whereas 4% rated candidate b with the same highest rating. For both candidates, the grading framework yields the following results:*

	6	**5**	**4**	**3**	**2**	**1**
a	21%	29%	50%	0%	0%	0%
b	4%	47%	2%	5%	7%	35%

The MJ says that *b* wins. All voters, however, awarded *a* a minimum of the score of 4, whereas over one-third (35%) rejected *b*. The optimal course of action is as suggested by RV, which favors *a* (average 4.71), to the detriment of *b* (average 3.31). This example shows that MJ fails to find the consensual candidate.

Example 4. *This instance is adapted from [26]. Refer to Example 3 for the same evaluation context:*

	6	**5**	**4**	**3**	**2**	**1**
a	2%	1%	97%	0%	0%	0%
b	31%	20%	0%	0%	0%	49%

Despite the fact that *b* is rejected by 49% of voters and *a* receives at least a grade 4 from all voters, the MJ asserts that *b* wins. RV circumvents this paradox by saying that *b* wins based on its average. In fact, the average for *a* is 4.05, whereas the average for *b* is 3.35. As with the last scenario, this one serves as an example of how MJ sometimes fails to reach a consensus.

Example 5. *Another example that is similar to the precedent is adapted from [18,26].*

	6	**5**	**4**	**3**	**2**	**1**
a	50%	0%	0%	0%	0%	50%
b	0%	0%	0%	0%	52%	48%

MJ claims that *b* prevails since 50% of voters give *a* the highest possible grade and all voters think *b* should receive no more than grade 2 (with 48% of the lowest marks). In this instance, the RV is chosen over the MJ since the average of *a* (3.5) is higher than the average of *b* (1.52).

Examples 1–5 demonstrate that there are situations in which RV's decision is preferable to MJ's, and vice versa. For a comprehensive discussion on how to select a central tendency metric, see [27]. As we will see later, it is possible to find a compromise method that will allow the best option to be selected in each of the given scenarios.

4. A New Description of the Mean-Median Compromise Method

In the previous section, we presented situations where range voting decisions are preferable to those of majority judgment, and vice versa. These examples illustrate the strengths and weaknesses of both methods. In this section, we claim that there exists a method capable of resolving these examples without exhibiting the pathologies seen in RV and MJ. This method must demonstrate robustness, a consensual nature, freedom from the tyranny of the majority pathology, and other democratic properties as well.

4.1. Mean-Median Compromise Method for Small Electorate

Let S be the set of candidates, n be the number of voters, and g_{ij} be the grade assigned by voter v_j to candidate c_i. The MMCM computes the final score $f(c_i)$ for each candidate c_i as follows:

- Step 1: Ordering of grades. Arrange the grades assigned to each candidate c_i in descending order: $g^*_{i1} \geq g^*_{i2} \geq \dots \geq g^*_{in}$
- Step 2: Determination of intermediate grades.
  - 2.1. Choose a degree of division k
  - 2.2. Compute the value of the amplitude of division $\varphi = \dfrac{n}{2^k}$
  - 2.3. Determine $2^k - 1$ grades such that
    - * For each value of p from 1 to $2^k - 1$
      - · Compute $r(p) = \lceil \varphi \times p \rceil$ where $\lceil x \rceil$ denotes rounding up to the nearest integer.
      - · If $r(p)$ is not equal to any previous $r(q)$ for $q < p$, then select the grade with rank $r(p)$ in the ordered set of grades as *intermediate grade*.
    - * Let $\mathcal{M}_i$ be the set of intermediate grades for candidate c_i, where $\mathcal{M}_i = \{g^*_{r_{l1}}, g^*_{r_{l2}}, \dots, g^*_{r_{l(2^k-1)}}\}$.
- Step 3: Computation of final score. Compute the arithmetic mean of intermediate grades for each candidate.

$$f(c_i) = \frac{1}{|\mathcal{M}_i|} \sum_{g \in \mathcal{M}_i} g \tag{1}$$

Let us consider six candidates evaluated by six judges, as defined in Table 4. The ordered profile is given in Table 5. The MMCM score with $k = 2$ for candidate C is computed as follows:

1. Ordered grades: [5, 5, 4, 4, 3, 2].
2. Degree of division: $\varphi = \dfrac{6}{2^2} = 1.5$.
3. Intermediate grades: for $p = 1, \dots, 2^2 - 1 = 3$.
   - $r(1) = \lceil 1.5 \times 1 \rceil = \lceil 1.5 \rceil = 2$
     The grade with rank 2 in the ordered set is 5.
   - $r(2) = \lceil 1.5 \times 2 \rceil = \lceil 3 \rceil = 3$
     The grade with rank 3 in the ordered set is 4.
   - $r(3) = \lceil 1.5 \times 3 \rceil = \lceil 4.5 \rceil = 5$
     The grade with rank 5 in the ordered set is 3.

   The intermediate grades are [5, 4, 3].
4. The MMCM score for candidate c is $\dfrac{5+4+3}{3} = 4$.

   If $k = 3$, the MMCM score for candidate c is computed as follows:

1. Ordered grades: [5, 5, 4, 4, 3, 2].
2. Degree of division: $\varphi = \dfrac{6}{2^3} = 0.75$.
3. Intermediate grades: for $p = 1, \dots, 2^3 - 1 = 7$.
   - $r(1) = \lceil 0.75 \times 1 \rceil = \lceil 0.75 \rceil = 1$
     The grade with rank 1 in the ordered set is 5.
   - $r(2) = \lceil 0.75 \times 2 \rceil = \lceil 1.5 \rceil = 2$
     The grade with rank 2 in the ordered set is 5.
   - $r(3) = \lceil 0.75 \times 3 \rceil = \lceil 2.25 \rceil = 3$
     The grade with rank 3 in the ordered set is 4.
   - $r(4) = \lceil 0.75 \times 4 \rceil = \lceil 3 \rceil = 3$
     Rank 3 has been previously found.

- $r(5) = \lceil 0.75 \times 5 \rceil = \lceil 3.75 \rceil = 4$
  The grade with rank 4 in the ordered set is 4.
- $r(6) = \lceil 0.75 \times 6 \rceil = \lceil 4.5 \rceil = 5$
  The grade with rank 5 in the ordered set is 3.
- $r(7) = \lceil 0.75 \times 7 \rceil = \lceil 5.25 \rceil = 6$
  The grade with rank 6 in the ordered set is 2.

The intermediate grades are [5, 5, 4, 4, 3, 2].

4. The MMCM score for candidate C is $\dfrac{5+5+4+4+3+2}{7} = 3.28571$.

Table 4. Example of profile.

	J_1	J_2	J_3	J_4	J_5	J_6
A:	5	5	4	5	5	5
B:	5	4	4	4	3	4
C:	2	5	3	4	4	5
D:	4	3	2	3	3	3
E:	3	2	4	3	3	3
F:	4	2	1	2	2	3

Table 5. Example of a profile arranged in descending order.

A:	5	5	5	5	5	4
B:	5	4	4	4	4	3
C:	5	5	4	4	3	2
D:	4	3	3	3	3	2
E:	4	3	3	3	3	2
F:	4	3	2	2	2	1

4.2. Mean-Median Compromise Method with Many Voters

When multiple voters participate, the grades they assign to each candidate are aggregated, and the frequency of each grade is expressed as a percentage. Consequently, the total electorate is normalized to 100%. Intermediate grades are then computed as follows:

$$\varphi = \frac{100}{2^k}$$

For $k = 2$, for example, intermediate grades are those with cumulative frequencies at 25%, 50%, and 75%. For $k = 3$, they are at 12.5%, 25%, 37.5%, 50%, 62.5 75%, and 87.5%.

Table 6 is a modified version of the example given in ([6], p. 440) based on a survey conducted by OpinionWay between April 12 and 16, 2012, a few days prior to the French presidential election's first round [21].

Table 6. OpiononWay result from 2012 French presidential poll (737 ballots) (data from [6]).

Candidates	**7**	**6**	**5**	**4**	**3**	**2**	**1**
F. Hollande	12.48	16.15	16.42	11.67	14.79	14.25	14.24
F. Bayrou	2.58	9.77	21.71	25.24	20.08	11.94	08.69
N. Sarkozy	9.63	12.35	16.28	10.99	11.13	7.87	31.75
J.-L. Mélenchon	5.43	9.50	12.89	14.65	17.10	15.06	25.37
N. Dupont-Aignan	0.54	2.58	5.97	11.26	20.22	25.51	33.92
E. Joly	0.81	2.99	6.51	11.80	14.65	24.69	38.53
P. Poutou	0.14	1.36	4.48	7.73	12.48	28.09	45.73
M. Le Pen	5.97	7.33	9.50	9.36	13.98	6.24	47.63
N. Arthaud	0.00	1.36	3.80	6.51	13.16	25.24	49.93
J. Cheminade	0.41	0.81	2.44	5.83	11.67	26.87	51.97

The MMCM is then applied as if there were only 100 voters. If, for instance, $k = 2$, the intermediate grades are those that are, respectively, ranked at 25%, 50%, and 75%, that is, the first three quartiles. Hence, the MMCM scores for F. Hollande and F. Bayrou are both 4. In fact, their intermediate grades are [6, 4, 2] for F. Hollande and [5, 4, 3] for F. Bayrou.

4.3. General Algorithm with Tie-Breaking Mechanism

The candidate with the better score naturally ranks ahead of the other when their MMCM scores differ. However, the following guidelines from [14,15] are applicable if both candidates have the same MMCM score.

Selecting the value of k is a crucial issue when using MMCM, as any value of k is permissible. If k is too large, at least one intermediate grade may be duplicated. This scenario is excluded by the method's definition (see Step 2.3 in Section 4.1). However, to avoid paradoxes found in MJ, we recommend starting with $k = 2$ for full-scale elections. The smallest value of k that equates the set of grades to the set of intermediate grades is termed the "index of the maximal division" and is denoted ν. This value is the upper bound of k in the algorithm (see Algorithm 1).

Algorithm 1: General MMCM algorithm

```
Begin with k = 2
Compute ν
Compute f^k(a) and f^k(b)
while f^k(a) = f^k(b) do
    k = k + 1
    Compute f^k(a) and f^k(b)
    if k = ν then
        exit
if f^k(a) > f^k(b) then
    a ranks ahead of b
else
    if f^k(b) > f^k(a) then
        b ranks ahead of a
    else
        a and b are tied as both have all the same grades.
```

5. Analyzing the MMCM

5.1. Index of the Maximal Division

Let $G = \{g_{i1}, g_{i2}, \ldots, g_{in}\}$ be the set of grades allotted by n voters to candidate c_i and $\mathcal{M}$ be the set of intermediate grades. The index of the maximal division, denoted ν, is the smallest degree of division k such that $\mathcal{M} = G$.

Theorem 1. *Given N as a set of n judges, the index of maximal division is determined by* $\nu = \lceil \log_2 n \rceil$.

Proof. We know that the division is maximal when $\mathcal{M} = G$. To conduct this, we must have

$$\frac{1}{2} \leq \varphi \leq 1$$

Hence,

$$\frac{1}{2} \leq \frac{n}{2^k} \leq 1$$

If we invert fractions, inequalities become

$$1 \leq \frac{2^k}{n} \leq 2$$

$$n \leq 2^k \leq 2n$$

$$\log_2 n \leq k \leq 1 + \log_2 n$$

From an Archimedean result, we obtain $k = \lceil \log_2 n \rceil$. □

The index of the maximal division ν ensures that the number of divisions grows logarithmically with the size of the electorate. The blue curve in Figure 1 shows the evolution of the index of maximal division as the number of voters increases. Even for billions of voters, the tie-breaking mechanism cannot exceed 40 iterations (See Figure 1).

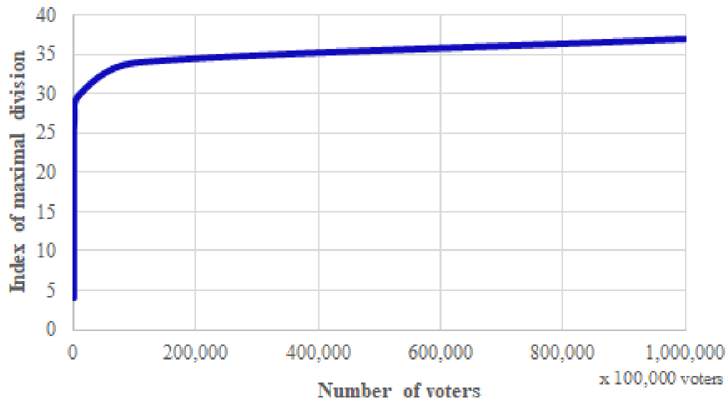

Figure 1. Growth of the index of maximal division.

5.2. Middlemost Aggregation

A middlemost is an aggregation function that identifies the middle value(s) in a set of ordered evaluations. If the number of evaluations is odd, the middlemost value is exactly the median. If the number of evaluations is even, the middlemost value is one of the two central values or an intermediate value of them (high middlemost and low middlemost). Concretely, a middlemost aggregation function f, for $g_1 \geq \cdots \geq g_n$ is $f(r_1, \ldots, r_n) = r_{(n+1)/2}$ when n is odd, and $f(r_1, \ldots, r_n) = r_{(n+2)/2}$ when n is even.

In [4], the authors advocated for the middlemost to be the unique aggregation functions that agree with the majority of judges in assigning a grade g that reduces the possibility of effective-manipulability or counter crankiness and maximizes welfare. As will be shown, when $k = 1$, MMCM is middlemost.

Theorem 2. *When the degree of division $k = 1$, the Mean-Median Compromise Method is a middlemost aggregation function.*

Proof. Let a be a candidate and $G = \{g_1^*, \ldots, g_n^*\}$ be the set of his or her grades with $g_1^* \leq \cdots \leq g_n^*$. If $k = 1$, then $\varphi = \dfrac{n}{2}$ and the unique intermediate grade is the one that ranks at $r(\lceil \frac{n}{2} \rceil)$, i.e., $g^*_{\lceil \frac{n}{2} \rceil}$. When n is odd $g^*_{\lceil \frac{n}{2} \rceil} = g^*_{\frac{n+1}{2}}$. When n is even, then $g^*_{\lceil \frac{n}{2} \rceil} = g^*_{\frac{n}{2}}$ is the upper-middlemost. Whatever the parity of n, the Mean-Median Compromise Method returns the middlemost. □

Theorem 3. *When the degree of division $k = \nu$, the Mean-Median Compromise Method is equivalent to the arithmetic mean.*

Proof. Let a be a candidate and $G = \{g_1^*, \ldots, g_n^*\}$ be the set of his or her grades with $g_1^* \leq \cdots \leq g_n^*$. If $k = \nu$, then $\mathcal{M} = G$. Hence, $f(a) = \dfrac{1}{|\mathcal{M}|} \sum_{g^* \in \mathcal{M}} g^* = \frac{1}{n} \sum_{i=1}^{n} g_i^*$. The last equality shows that, when $k = \nu$, the MMCM is the definition of arithmetic mean. □

5.3. Desired Properties

It is checked that the Mean-Median Compromise Method possesses, *inter alia*, these properties (see Theorem 4 below):

- Anonymity: This property ensures that all voters are treated equally. If any two voters were to swap their ballots, the result would remain unchanged. Whenever voters are permuted, the MMCM does not change the winner. Aggregation is based on merits and not people who assess them (see Section 2.3 for mathematical definition).
- Neutrality: This property ensures that the voting method treats all candidates equally. In other words, if any two candidates were to swap places in every voter's ballot, the result would still reflect that swapped order. The MMCM treats all candidates equally in the calculation of the final score (see Section 2.3 for mathematical definition).
- Pareto effeciency: Given a set of candidates $C = \{c_1, c_2, \ldots, c_m\}$ and a set of voters $V = \{v_1, v_2, \ldots, v_n\}$, each voter assigns a grade g_{ij} to each candidate c_j. A grading mechanism f is Pareto-efficient if, for any two candidates c_x and c_y,

  $$\forall v_i \in V,\ g_{ix} \geq g_{iy} \Rightarrow MMCM(c_x) \geq MMCM(c_y)$$

  where $MMCM(c)$ denotes the MMCM score for candidate c.
- Independence of irrelevant alternatives: Consider $C = \{c_1, \ldots, c_m\}$ a set of m candidates. If for any two candidates $c_x, c_y \in C$, the overall community preference is, say, $c_x \succ c_y$, then this ranking should not be affected by the addition or removal of any other candidate c_z, whatever grades voters assigned to them. The MMCM score depends only on the grades assigned to the candidates, not on the presence or absence of the other candidates.
- Monotonicity: Let $V = \{v_1, \ldots, v_n\}$ be the set of voters, and $C = \{c_1, \ldots, c_m\}$ the set of m candidates. Let $G = \{g_{1j}, \ldots, g_{nj}\}$ be the set of grades assigned by voters v_i to a fixed candidate c_j. Suppose the MMCM score with respect to G is s. If any grade g_{ij} (for $i = 1, \ldots, n$) is replaced by g'_{ij} such that $g'_{ij} \geq g_{ij}$, we denote the new MMCM score as s'. Then, it must hold that $s' \geq s$.
- Robustness to strategic voting: MMCM is robust against strategic voting, meaning that voters cannot manipulate the outcome by strategically misrepresenting their preferences. This property ensures that the final result reflects the genuine preferences of the voters and prevents individuals from strategically gaming the system to their advantage. The robustness to strategic voting is essential for ensuring the integrity and fairness of the voting process. Computer simulations will show that MMCM is relatively strategy-proof.
- Consensus building: MMCM fosters consensus building by promoting compromise and accomodating a broad range of voter preferences. This property encourages candidates to appeal to a broader base of voters and discourages polarization, ultimately contributing to the overall stability and cohesion of the voting system. An interesting and detailed discussion on polarization is provided by ([6], pp. 448–452).

Theorem 4. *MMCM meets neutrality, anonymity, Pareto efficiency, monotonicity, and independence of irrelevant alternative properties.*

Proof. To show that MMCM is neutral, we consider an electorate $N = \{v_1, v_2, \ldots, v_n\}$ of n voters ($n \geq 2$). If two candidates c_i and c_j are evaluated by these n voters and their grades are, respectively, $G_i = \{g_{1i}, g_{2i}, \ldots, g_{ni}\}$ and $G_j = \{g_{1j}, g_{2j}, \ldots, g_{nj}\}$, where $g_{x\alpha}$ indicates the grade allotted by voter x to candidate $\alpha (x = 1, \ldots, n; \alpha = i \text{ or } j)$. Suppose that $f(c_i)$ is the MMCM score for candidate c_i. If $f(c_i) > f(c_j)$, then $c_i \succ_{mm} c_j$ (read "c_i

ranks ahead of c_j by MMCM"). If for each voter $v_x \quad (x = 1, \ldots, n)$ g_{x_i} becomes g_{x_j} and vice versa, then the set of i's grades becomes $G_j = \{g_{1j}, g_{2j}, \ldots, g_{nj}\}$ and j's one becomes $G_i = \{g_{1i}, g_{2i}, \ldots, g_{ni}\}$. Therefore, $f(c_j) > f(c_i) \Rightarrow c_j \succ_{mm} c_i$ and MMCM is neutral.

Now, with the set of grades G_i for candidate c_i, if voters v_x and v_y permute (i.e., g_{xi} becomes g_{yi} and vice versa whatever $x, y = 1, \ldots, n$), G_i will not change. Thus, $f(c_i)$ will not change even if the voters were permuted. As c_i is unspecified, this remains true for any candidate, showing that MMCM is anonymous.

Let c_i and c_j be two candidates with perspectives grades $G_i = \{g_{1i}, g_{2i}, \ldots, g_{ni}\}$ and $G_j = \{g_{1j}, g_{2j}, \ldots, g_{nj}\}$, such that $g_{xi} > g_{xj} \quad \forall 1 \leq x \leq n$. We will obtain for any division degree k (with $k \geq 1$): $\mathcal{M}_i = \{{g^\star}_{1i}, {g^\star}_{2i}, \ldots, {g^\star}_{mi}\}$ and $\mathcal{M}_j = \left\{{g^\star}_{1j}, {g^\star}_{2j}, \ldots, {g^\star}_{mj}\right\}$ (with $m = 2^k - 1$) where $\mathcal{M}_x$ indicates the set of intermediate grades for candidate c_x. Since $g_{xi} > g_{xj} \quad \forall 1 \leq x \leq n$, we have ${g^\star}_{xi} > {g^\star}_{xj} \quad \forall 1 \leq x \leq m$, and thus,

$$\frac{1}{m}\sum_{i=1}^{m} {g^\star}_{xi} > \frac{1}{m}\sum_{i=1}^{m} {g^\star}_{xj} \Rightarrow f(c_i) > f(c_j) \Rightarrow c_i >_{mm} c_j.$$

This last expression establishes that MMCM meets the Pareto efficiency property.

To show the monotonicity of MMCM, we consider two candidates c_i and c_j with perspective grades G_i and G_j, such that $f(c_i) > f(c_j)$, i.e., $c_i \succ_{mm} c_j$. Suppose that voter v_x, having previously allotted grade g_{xi} to c_i, re-evaluated her or him by allotting a grade ${g'}_{xi}$ such that ${g'}_{xi} > g_{xi}$ *ceteris paribus*. Three cases are then possible, as follows:

- Grade ${g'}_{xi}$ does not amend the overall constitution of intermediate grades $\mathcal{M}_i$. Then, $f(c_i)$ remains the same and $f(c_i) > f(c_j)$.
- Grade ${g'}_{xi}$ is an intermediate grade (i.e., voter v_x is pivotal) and $f_x(c_i)$ is c_i's final evaluation by MMCM after taking into account the preference amendment of voter v_x. Thus $f_x(c_i) > f(c_i) > f(c_j) \Rightarrow f_x(c_i) > f(c_j)$.
- Grade ${g'}_{xi}$ is not an intermediate grade but amends the overall constitution of intermediate grades $\mathcal{M}_i$. In this case, an intermediate grade is replaced by another by shifting a row on the left. Let ${g^\star}_{li}$ $(1 \leq l \leq 2^k - 1)$ be the replaced grade. This grade is replaced by ${g^\star}_{(l-1)i}$. However, ${g^\star}_{(l-1)i} \geq {g^\star}_{li}$ (grades are arranged in decreasing order prior to the evaluation of f). We then have $f_x(c_i) \geq f(c_i) > f(c_j) \Rightarrow f_x(c_i) > f(c_j)$.

It is demonstrated that MMCM is monotonic through these three potential scenarios.

Finally, we show that MMCM is independent of irrelevant alternatives. Voters base their assessments on the performance of each candidate, regardless of one another. Therefore, if a voter v_x assigns grade g_{xi} to candidate c_i and g_{xj} to another candidate c_j so that $g_{xi} \geq g_{xj}$, then whatever grade g_{xt} she or he assigns in addition to candidate c_t, the order $g_{xi} \geq g_{xj}$ will never be changed. This clearly shows that MMCM is independent of irrelevant alternatives and concludes our demonstration. □

5.4. Homogeneity Property

The homogeneity property in the context of social choice functions states that if all individual preferences or evaluations are multiplied by the same factor, the overall preference or evaluation order should remain unchanged. In [28], it is shown that the MMCM, in its former formulation, is not homogeneous. As will be shown, the new formulation of the Mean-Median Compromise Method makes it meet this condition.

Theorem 5. *The Mean-Median Compromise Method meets the homogeneity condition.*

Proof. To prove this theorem, we need to show that the MMCM score remains unchanged when each grade is replicated p times $(p \geq 2)$. Let us denote the original grades allotted by voters to candidates i as g^*_{ij} (with $j = 1, \ldots, n$), and let p be the replication factor. The original and replicated sets of grades are, respectively, given by decreasing ordered sets

$$G_o = \{g^*_{i1}, \ldots, g^*_{in}\} \text{ and } G_r = \{\underbrace{g^*_{i1}, \ldots g^*_{i1}}_{p \text{ times}}, \ldots, \underbrace{g^*_{in}, \ldots g^*_{in}}_{p \text{ times}}\}$$

A grade $g^*_{ij} \in G_o$ replicated p times occupies ranks $(j-1) \times p + 1$ to $i \times p$ in G_r. For a degree of division k, we have exactly $2^k - 1$ intermediate grades rankings $r_l = \lceil \frac{n \times l}{2^k} \rceil$ in G_o $(l = 1, \ldots, 2^k - 1)$. In the case voters are replicated p times, intermediate grades rakings should be $r^*_l = \lceil \frac{n \times l}{2^k} \times p \rceil$. It suffices for us to show that $r^*_l \in I_l = [(r_l - 1) \times p + 1, r_l \times p]$ $\forall \quad l = 1, \ldots, 2^k - 1$, i.e.,

$$\lceil \frac{n \times l}{2^k} \rceil \times p - p + 1 \leq \lceil \frac{n \times l}{2^k} \times p \rceil \leq \lceil \frac{n \times l}{2^k} \rceil \times p \tag{2}$$

We already know that $\lceil x \times y \rceil \leq \lceil x \rceil \times \lceil y \rceil \quad \forall \quad x, y \geq 0$. The second inequality of Equation (2) is then checked. According to Theorem 1, the division is total if $\frac{1}{2} \leq \varphi = \frac{n}{2^k} \leq 1$. At worst case, if $\theta = \frac{n \times l}{2^k} - \mathcal{I}(\frac{n \times l}{2^k})$, where $\mathcal{I}(x)$ denotes the integer part of real number x, we have $\lceil \frac{n \times l}{2^k} \rceil \leq \mathcal{I}(\frac{n \times l}{2^k}) + 1$ and $\lceil \frac{n \times l}{2^k} \times p \rceil \leq \mathcal{I}(\frac{n \times l}{2^k} \times p) + 1 \leq \left[\mathcal{I}(\frac{n \times l}{2^k} \times p) + 1\right] p - p + 1 = \mathcal{I}(\frac{n \times l}{2^k}) \times p + 1$ (i). We can easily check the middle expression of Equation (2), which can be expressed as $\mathcal{I}(\frac{n \times l}{2^k}) \times p + \theta \times p$ (ii). From (i) and (ii), we deduce the first inequality of Equation (2). $\square$

5.5. Tyranny of Majority

The "Kill the Jews" vote is an example of the kind of majority dictatorship that [29] discusses. The vote is framed as a choice between "letting them live on their wealth" and "killing the Jews and using their money to cut taxes for the survivors". A comparable referendum would be held in a society (in Africa, for instance) to decide whether to execute or not to execute LGBTQIA+ individuals.

Certain vote outcomes may benefit the majority marginally while having extremely negative impacts on the minority. Assume that there are only two types of votes in this society, with a rating scale of 0 to 9. These are the strong preference votes, "Kill = 0, Live = 9", expressed by LGBTQIA+ individuals, and "Kill = 5, Live = 4", expressed by non-LGBTQIA+ individuals. In such a case, RV will permit LGBTQIA+ people to remain alive if their percentage of voters exceeds 10%. But as long as LGBTQIA+ people remain a minority, which is frequently the situation in many communities, particularly in Africa, MJ will kill them regardless of their percentage.

We can challenge this minority with a strong preference that forces its choice on society as a whole if we thoroughly examine this issue. Is the opinion of a minority of 10% truly appropriate for the entire community to follow?

By allowing a reasonable minority with strong preferences to override a majority with weak preferences, MMCM lessens the effects of the tyranny of the majority. MMCM allows LGBTQIA+ people to live if their percentage is at least 25%. It is evident that MMCM is majority-tyranny-proof (which MJ is not) and robust (which RV is not).

As a general rule, if the valuation scale is $[\alpha, \beta]$, the minority and majority ratios are n_1 and n_2, respectively. Minorities have strong preferences over the alternatives "Kill the Minority" and "Save the minority", to which they allot α and β ratings, respectively. The majority have a weak preference for these two alternatives, allotting them ω_1 and ω_2, respectively, (with $\omega_1 > \omega_2$). Under these conditions, the following theorems are established:

Theorem 6. *Subject to the aforementioned conditions, minorities will be saved, using range voting, if* $n_1 > \dfrac{\omega_1 - \omega_2}{\beta - \alpha + \omega_1 - \omega_2}$.

Proof. To overcome the majority, the minority must score higher on the range voting for "Save minority" than for "Kill minority", that is,

$$\beta n_1 + \omega_2 n_2 > \alpha n_1 + \omega_1 n_2 \quad (*)$$

Given that $n_1 + n_2 = 1$, $n_2 = 1 - n_1$ (**) follows. We obtain the following if we replace (**) in (*): $\beta n_1 + \omega_2(1 - n_1) > \alpha n_1 + \omega_1(1 - n_1)$. We determine $n_1 > \dfrac{\omega_1 - \omega_2}{\beta - \alpha + \omega_1 - \omega_2}$ by developing and solving this inequation with regard to n_1. □

Theorem 7. *By MMCM, it takes* $\beta > 3\omega_1 - 2\omega_2$ *for the 25% minority to defeat the 75% majority.*

Proof. We can easily check that $[\beta, \omega_2, \omega_2]$ and $[\omega_1, \omega_1, \omega_1]$ are the intermediate grades for "Save the Minority" and "Kill the Minority", respectively. Therefore, $\frac{\beta+\omega_2+\omega_2}{3} > \frac{\omega_1+\omega_1+\omega_1}{3}$ (***) is required for "Save the Minority" to prevail. With regard to β, calculating the equation (***) yields the following result: $\beta > 3\omega_1 - 2\omega_2$. □

5.6. Computational Complexity

We break down the computional complexity of MMCM into its main steps, as follows:

1. Sorting: Sorting the grades in descending order has a time complexity of $\mathcal{O}(n \log n)$, where n is the number of grades.
2. Intermediate grade computation:
   - Determining the degree of division k and computing the values of φ have a constant time complexity $\mathcal{O}(1)$.
   - Computing the ranks $r(p)$ involves multiplying $\varphi \times p$ and rounding up to the nearest integer. This operation is performed $2^k - 1$ times and has a time complexity of $\mathcal{O}(2^k)$.
   - Finding the corresponding grade for each $r(p)$ involves accessing elements in the ordered list of grades. This operation is performed $2^k - 1$ times and has a time complexity of $\mathcal{O}(1)$ for each access.
3. Final score: Summing the intermediate grades and dividing by the number of intermediate grades has a time complexity of $\mathcal{O}(2^k)$, as it involves summing $2^k - 1$ grades.

Overall, the dominant step in the MMCM algorithm is the computation of intermediate grades, which has a time complexity of $\mathcal{O}(2^k)$. The degree of division is at most $\nu = \lceil \log_2 n \rceil$, so the overall time complexity of MMCM at worst case is $\mathcal{O}(2^{\lceil \log_2 n \rceil}) = \mathcal{O}(n)$. In summary, the MMCM has a time complexity of $\mathcal{O}(n)$ as it operates efficiently even for large numbers of grades.

5.7. Other Democratic Properties in the Literature

In this paragraph, we list and define more widely accepted democratic standards for a voting function within the context of social choice theory. Since there is a large collection of these attributes, the list is not all-inclusive.

- Universality: A fair voting function must ensure that each person participating in the decision-making process has a certain amount of freedom. Furthermore, while determining the election winner, each of their choices needs to be considered.
- Non-dictatorship: A fair voting mechanism must avoid a dictator. A voter v_i is called a dictator if, when they assign candidate c a rating higher than any other candidate, regardless of the overall profile, candidate c is always the winner.

- Resoluteness: The function must always select the winner among the m candidates $c_1, c_2, \ldots, c_m$.
- Choice-monotonicity: A voting rule is choice-monotonic if $a \succ b$, and, in the assessments of some or all voters, candidate a moves strictly higher or candidate b moves strictly lower, then it still returns $a \succ b$.
- Rank-monotonicity: In addition to the winner staying in first place, the final ranking among the other candidates should also stay the same if voters' assessments stay the same but the winner moves up.
- Strong monotonic: In the event that the public's opinion of a non-winner declines, the winner remains the front-runner.
- Clone-resistance: The introduction of a clone of a losing alternative never modifies the initial result.
- Expressiveness: If a voting rule asks voters for additional details about their preferences—such as completeness, intensity, etc.—it is considered expressive.
- Reinforcement condition: If an electorate is split into two groups and the results of the votes are the same for each group, then the results will not change when the two voter groups are combined.
- Participation condition: If adding some voters who favor a winning candidate does not make that candidate lose, then the decision rule is said to satisfy the Participation requirement.
- Majority condition: A decision rule satisfies the majority condition if candidate a is consistently delivered as the winner whenever an absolute majority rates candidate a higher than candidate b.
- Condorcet winner condition: If a decision rule consistently chooses the Condorcet winner when one exists, it is said to satisfy the Condorcet winner criterion. A candidate who receives the utmost support from the electorate over all others is declared the Condorcet winner.
- Condorcet Loser condition: If a decision rule never chooses the Condorcet loser when one exists, it is said to satisfy the Condorcet loser condition. A candidate who loses a one-on-one match to any other contender is considered a Condorcet loser.
- Transitivity condition: When the electorate's preferences are $a \succ b$ and $b \succ c$, given three possibilities $a, b, c \in A$, then we must have $a \succ c$. This is known as a transitive decision rule. The Condorcet paradox must be avoided in order to meet this requirement.
- Simpson's condition: When a decision rule does not result in the Simpson's paradox—a phenomenon wherein, upon merging two distinct groups, an apparent relationship seems to be reversed—it is said to satisfy Simpson's condition.
- Polynomial runtime complexity: If a decision rule converges to result in polynomial time—that is, in a reasonable amount of time—it is said to be runtime polynomial time.

5.8. Democratic Properties Not Satisfied by MMCM

It is demonstrated in [12] that a few fundamental paradoxes, such as Condorcet's and no-show paradoxes, arise at L^p depths, including the MMCM. However, refs. [3,12] discuss these shortcomings in relation to the grading framework. The MMCM is essentially a variant of deepest voting, wherein specific depth functions serve as tie-breaking mechanisms.

6. Characterization of the MMCM

As previously shown, the MMCM is a L^p deepest and method of grading, which satisfies multiple democratic requirements. What interests us in this case is compiling a short list of properties that are exclusively filled by MMCM. Let us first define the terms as follows:

Definition 1 (Pivotal function)**.** *Let* $\tau = \{g_1, \ldots, g_n\}$ *be a distribution and* $f : \mathbb{R}^n \to \mathbb{R} : (g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_n)$ *be a method of grading. We say that* f *is a "pivotal function" if*

$\exists P_i = \{\tilde{g}_1, \ldots, \tilde{g}_l\} \subsetneq \tau$ *and a numerical function* $\psi : \mathbb{R}^l \to \mathbb{R}$ *such that* $f(g_1, \ldots, g_n) = \psi(\tilde{g}_1, \ldots, \tilde{g}_l)$. *The function* ψ *is called the "characterizer of* f*".*

Since only the intermediate grades are used to determine the final score, MMCM is unquestionably pivotal. The method's pivots are the intermediate grades. Observe that altering the data size impacts every intermediate grade, ensuring the method's universality.

Definition 2 (Regularity)**.** *Let* $(g_1, \ldots, g_n) \in \mathbb{R}^n$ *be a grade distribution. We consider the pivotal function* $f : \mathbb{R}^n \to \mathbb{R}$, *whose pivots are* $\tilde{g}_1, \ldots, \tilde{g}_l$. *Assume that* k *bipartition of the distribution yields intervals* $I_p \in \{g_1, \ldots, g_n\}$, *where* $p = 1, \ldots, 2^k - 1$. *When there is a pivot* $\tilde{g}_t$ *such that* $\tilde{g}_t = f^m(I_p)$, *where* f^m *is a middlemost aggregation function, then the pivot is said to be "regular". If every pivot in* f *is regular, then* f *is referred to as a "regular function".*

Definition 3 (Mean-wise function)**.** *Let* f *be a pivotal function. We say that* f *is a "mean-wise function" if its characterizer* ψ *is a mean function.*

With the definitions given, we can state the characterization theorem of the MMCM as follows:

Theorem 8. *The MMCM is the only method of grading that is simultaneously pivotal, regular, and mean-wise allowing the arithmetic mean as characterizer.*

Proof. Let $\tau = \{g_1, \ldots, g_n\}$ be a set of grades and $f : \mathbb{R}^n \to \mathbb{R}$ be a method of grading. Since f is pivotal and regular, there exists $P_i = \{\tilde{g}_1, \ldots, \tilde{g}_l\} \subsetneq \tau$ and $\psi : \mathbb{R}^l \to \mathbb{R}$ such that $f(g_1, \ldots, g_n) = \psi(\tilde{g}_1, \ldots, \tilde{g}_l)$. The characterizer of f is the arithmetic mean. So,

$$f(g_1, \ldots, g_n) = \psi(\tilde{g}_1, \ldots, \tilde{g}_l) = \frac{1}{l}\sum_{i=1}^{l} \tilde{g}_i$$

The last expression is the definition of MMCM. □

7. Comparing MMCM with Alternative Voting Mechanisms

The democratic properties desired for a voting function are as numerous as they are diverse. In this section, we make a comparison between MMCM and the most discussed voting functions in the literature on the Theory of Social Choice. Some prior definitions of the various properties are given above. Table 7 summarizes these properties as well as the functions selected for comparison. The last column in gray is the one reserved for MMCM. When a cell has a "+" in it, the voting function on the column fills the row's property; if a cell has a "−", it does not.

MMCM is clearly the voting mechanism that meets the most democratic requirements, as evidenced by the fact that 19 out of 25 selected criteria are filled. While not exhaustive, this list does include the most often-mentioned criteria in the literature. The fact that the MMCM meets the majority of the most-discussed democratic criteria reassures us to defend it. Using MMCM in real-life situations could ensure that every vote counts and that everyone's voice is heard. Next, MMCM covers the aspects of equity and inclusion in decision-making processes.

Table 7. Voting rules with their satisfied and failed criteria.

Normative criteria	Plurality	Negative Plurality	Borda/Dowdall	Condorcet	Bucklin	Instant runoff	Coombs	Nanson	Baldwin	Dodgson	Copeland	Black	Simpson–Kramer	Ranked pairs	Smith	Schwartz	Schulze	Shapley ranking	Majority judgment	Range voting/BMC	MMCM
Universality	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Independence of irrelevant alternatives	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	+	+	+
Pareto	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Non-dictatorship	−	−	+	−	+	+	+	+	−	−	+	+	−	+	−	−	+	−	+	−	+
Resoluteness	+	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+
Neutrality	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Monotonicity	+	+	+	+	+	−	−	−	−	−	+	+	+	+	+	+	+	+	+	+	+
Anonymity	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Clone−resistance	−	−	−	+	−	+	−	−	−	−	−	−	−	+	−	−	+	+	+	+	+
Expressiveness	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Homogeneity	+	+	+	+	+	+	+	−	+	−	+	+	+	+	+	+	+	+	+	+	+
Reinforcement condition	+	+	+	+	+	+	−	−	+	−	+	+	−	+	−	−	−	+	−	+	−
Participation condition	+	+	+	+	−	−	−	−	−	−	−	+	−	+	−	−	−	−	−	+	−
Majority condition	−	−	−	+	+	+	+	+	+	+	+	−	+	+	+	+	+	+	−	−	−
Majority tyranny proofness	−	+	−	−	−	−	+	−	−	−	−	−	−	−	−	−	−	−	−	+	+
Condorcet winner condition	−	−	−	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	−	−	−
Condorcet loser condition	−	+	+	+	+	−	+	+	+	−	+	+	−	+	+	+	+	−	−	−	−
Transitivity condition	+	+	+	−	+	+	+	+	+	+	+	+	+	−	−	−	+	+	+	+	+
Simpson's condition	−	−	−	−	−	−	−	+	+	−	+	−	−	+	+	+	+	+	−	+	−
Strategy−proofness (*)	−	−	−	−	+	−	−	−	−	−	−	−	−	−	+	+	−	−	+	−	+
Polynomial runtime complexity	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
Consensus building (*)	−	−	−	−	+	−	−	−	−	−	−	−	−	−	+	+	−	−	+	−	+
Balanced outcome (*)	−	−	−	−	+	−	−	−	−	−	−	−	−	−	+	+	−	−	−	+	+
Moderation of extremes (*)	−	−	−	−	+	−	−	−	−	−	−	−	−	−	+	+	−	−	+	−	+
Convergence to social consensus (*)	−	−	−	−	+	−	−	−	−	−	−	−	−	−	+	+	−	−	+	−	+
Total of filled criteria	11	14	13	14	15	13	13	13	14	10	15	15	12	17	14	14	16	15	17	17	**19**

(+): criterion is filled by the voting rule; (−): criterion is not filled by the voting rule; (*) see discussions on this criterion in Section 9.

8. MMCM in Real-World Situations

8.1. The Judgment of Paris

Owner of the *Caves de la Madeleine* in Paris, Steven Spurrier, along with Patricia Gallagher of the French *Académie du Vin*, arranged a blind tasting of four wines each of white Burgundies and red Bordeauxs, as well as six white and six red Californian wines that were, at best, unknown and, at worst, disregarded in Europe. All eleven judges—sommeliers, wine journalists, makers of well-known wines, and proprietors of Michelin-starred restaurants—were incredibly skilled wine enthusiasts. Both the white wines (Chateau Montelena) and the red wines (Stag's Leap Wine Cellars) from California were graded higher than the French wines. As noticed by [30], this enhanced the standing of Californian wines and altered the conventional wisdom held by specialists that only French wines could be of exceptional quality. This is known as the "Judgment of Paris".

In this paper, we analyze the competition for both red and white wines, and we compare the final rankings achieved by MMCM with the results obtained using the most popularized voting rules. Before we show the results, let us first present the voters and the candidates. The list of competing red wines and Chardonnays is given in Table 8.

Table 8. Lists of competing red wines and Chardonnays (data from [31]).

Red wines			
Code	**Wine**	**Year**	**Origin**
A	Stag's Leap	1973	Californian
B	Château Mouton Rothschild	1970	French
C	Château Montrose	1970	French
D	Château Haut-Brion	1970	French
E	Ridge Vineyards Monte Bello Cabernet Sauvignon	1971	Californian
F	Château Léoville—Las Cases	1971	French
G	Heitz Wine Cellars Martha's Vineyard Cabernet Sauvignon	1970	Californian
H	Clos du Val Cabernet Sauvignon	1972	Californian
I	Mayacamas Vineyards Cabernet Sauvignon	1971	Californian
J	Freemark Abbey Winery Cabernet Sauvignon	1969	Californian
Chardonnays			
Code	**Wine**	**Year**	**Origin**
A	Chateau Montelena Chardonnay	1973	Californian
B	Mersault Charmes Roulot	1973	French
C	Chalone Vineyard Chardonnay	1974	Californian
D	Spring Mountain Vineyard Chardonnay	1973	Californian
E	Freemark Abbey Winery Chardonnay	1972	Californian
F	Bâtard-Montrachet Ramonet-Prudhon	1973	French
G	Puligny-Montrachet Les Pucelles Domaine Leflaive	1972	French
H	Beaune, Clos des Mouches Joseph Drouhin	1973	French
I	Veedercrest Vineyards Chardonnay	1972	Californian
J	David Bruce Winery Chardonnay	1973	Californian

The eleven judges were as follows, listed alphabetically:

- Pierre Brejoux (French) of the Institute of Appellations of Origin;
- Claude Dubois-Millot (French) (Substitute to Christian Millau);
- Michel Dovaz (French) of the Wine Institute of France;
- Patricia Gallagher (American) of l'*Academie du Vin*;
- Odette Kahn (French) Editor of *La Revue du vin de France*;
- Raymond Oliver (French) of the restaurant *Le Grand Véfour*;
- Steven Spurrier (British), owner of the *Caves de la Madeleine*;
- Pierre Tari (French) of Chateau Giscours;
- Christian Vanneque (French), the sommelier of *Tour d'Argent*;
- Aubert de Villaine (French) of the *Domaine de la Romanée-Conti*;

- Jean-Claude Vrinat (French) of the *Restaurant Taillevent*.

8.1.1. Preferences of Judges

Each wine was given a score out of 20 by the judges during the Paris blind testing. The judges were allowed to grade using their own standards because no set grading scheme was provided. Calculating the arithmetic mean of each judge's individual rating also resulted in an overall ranking of the wines that the jury preferred. Competing wines were ranked without consideration of Patricia Gallagher's or Steven Spurrier's grades. According to [32], only the French judges' grades were taken into account. But in order to determine the final result, the scores of Steven Spurrier and Patricia Gallagher were taken into account, according to [33]. In this paper, we agree with [31] to compute the final result with Gallagher and Spurrier. The judges' preferences are given in Table 9.

Table 9. Original grades of the judges for red wines and Chardonnays (data from [31]).

Red wines										
Judges/Wines	A	B	C	D	E	F	G	H	I	J
Pierre Brejoux	14	16	12	17	13	10	12	14	5	7
Claude Dubois-Millot	16	16	17	13.5	7	11	8	9	9.5	9
Michel Dovaz	10	15	11	12	12	10	11.5	11	8	15
Odette Kahn	15	12	12	12	7	12	2	2	13	5
Raymond Oliver	14	12	14	10	12	12	10	10	14	8
Pierre Tari	13	11	14	14	17	12	15	13	12	14
Christian Vanneque	16.5	16	11	17	15.5	8	10	16.5	3	6
Aubert de Villaine	15	14	16	15	9	10	7	5	12	7
Jean-Claude Vrinat	14	14	15	15	11	12	9	7	13	7
Patricia Gallagher	14	15	14	12	16	14	17	13	9	15
Steven Spurrier	14	14	14	8	14	12	13	11	9	13
White wines										
Judges/Wines	A	B	C	D	E	F	G	H	I	J
Pierre Brejoux	10	15	16	10	13	8	11	5	6	0
Claude Dubois-Millot	18.5	15	11	15	10	9	8	14	16	4
Michel Dovaz	3	12	16	10	4	10	5	4	7	2
Odette Kahn	16.5	16	12	10	13	9	8	16	5	1
Raymond Oliver	17	14	13	12	12	12	12	14.5	10	7
Pierre Tari	14	13	16	13	9	14	13	12	14	8
Christian Vanneque	16.5	16	14	9	15	7	9	6	8	5
Aubert de Villaine	18	15	10	13	12	13	10	15	12	8
Jean-Claude Vrinat	17	14.5	13	12	12	12	12	14.5	10	7
Patricia Gallagher	14	16	15	16	13	11	15	7	12	11
Steven Spurrier	14	11	14	10	15	15	14	7	10	10

8.1.2. Applying MMCM in the Paris 1976 Wine Tasting

Since the judges' preferences are known, we applied MMCM to the 1976 Paris wine tasting competition. As the electorate consists of 11 members, we have

- For $k = 2$, $\varphi = \dfrac{11}{2^2} = 2.75$, and the three intermediate grades are the grades occupying ranks 3, 6, and 9.
- For $k = 3$, $\varphi = \dfrac{11}{2^3} = 1.375$ and the seven intermediate grades are the grades occupying ranks 2, 3, 5, 6, 7, 9, and 10.

Table 10 shows the MMCM scores achieved by each wine based on the preferences of Table 9. The gray cells indicate ex aequo that required the tie-break rule.

Table 10. MMCM scores on the Paris 1976 wine tasting.

Red wines										
Wines	A	B	C	D	E	F	G	H	I	J
$k = 2$	14.33	14	13.66	13.5	12.16	11.33	10.33	10.33	10.16	9.66
$k = 3$	–	–	–	–	–	–	10.64	10.14	–	–
Ranking	1	2	3	4	5	6	7	8	9	10
Chardonnays										
Wines	A	B	C	D	E	F	G	H	I	J
$k = 2$	15.83	14.66	14	11.66	11.66	11	10.66	10.83	9.66	5.66
$k = 3$	–	–	–	11.71	12	–	–	–	–	–
Ranking	1	2	3	5	4	6	8	7	9	10

8.1.3. The Paris 1976 Wine Tasting Result

We exclusively concentrate on the voting rules that have received the most attention and discussion in the social choice literature (see Appendix A) out of the many voting rules that were specified in the literature (see, for example, [34,35]).

We use the voting rules specified herein to aggregate the judges' preferences from the 1976 Paris wine tasting in Tables 11 and 12. Preferences as stated here do not allow the application of the intriguing voting rule known as Shapley ranking, which was suggested by [30]. For further information, the reader is directed to [30]. Hereafter, we consider that if all m alternatives are tied at rank i, all of those m alternatives are ranked

$$k = \sum_{j=i}^{i+m-1} \frac{j}{m} = \frac{m-1}{2} + i$$

Next, in the ranking, the alternative that comes after will be ranked $(i + m)$. For instance, if four options are tied at rank three, they are all ranked $\frac{4-1}{2} + 3 = 4.5$.

Table 11. The Paris 1976 wine tasting: ranking red wines using different voting rules.

Wines / **Voting Rules**	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
					Rankings					
Range voting	1	2	3	4	5	6	7	8	9	10
Plurality	3	6	1	2	4	9.5	5	9.5	8	7
Negative Plurality	2	5.5	2	5.5	5.5	2	5.5	9	10	8
Borda	1	3	2	4	5	7	6	10	9	8
Dowdall	3	4	1	2	5	10	6	9	7	8
Bucklin	1	2	4	3	5	6.5	8	9.5	9.5	6.5
Instant Runoff Voting	5	9	3	1	6	4	2	10	8	7
Coombs	2	3.5	1	3.5	6	5	7	8	10	9
Nanson	2.5	2.5	1	4.5	4.5	8	8	8	8	8
Baldwin	2.5	2.5	4.5	4.5	1	7.5	7.5	10	7.5	7.5
Dodgson	2.5	4	1	2.5	5	7	6	10	9	8
Copeland	2	4	1	3	5	7	6	9	10	8
Black	1	3	2	4	5	7	6	10	9	8
Simpson–Kramer	3	3	1	3	5	7.5	7.5	7.5	10	7.5
Ranked pairs	2.5	4	1	2.5	5	7	6	9	10	8
Smith	3	3	3	3	3	7	6	9	10	8
Schwartz	2.5	4	1	2.5	5	7	6	9	10	8
Quandt	1	3	2	4	5	7	6	10	9	8
Majority judgment	2	1	3	4	5	6	8	7	9	10
Borda Majority Count	1	2	3	4	5	6	7	8	9	10
Mean-Median Compromise Method *	1	2	3	4	5	6	7	8	9	10

*: Ranking obtained after tie-breaking.

Table 12. The Paris 1976 wine tasting: ranking Chardonnays using different voting rules.

Wines	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Voting Rules					**Rankings**					
Range voting	1	2	3	4	5	6	7	8	9	10
Plurality	1	4.5	2	4.5	4.5	4.5	8.5	8.5	8.5	8.5
Negative Plurality	4.5	4.5	4.5	4.5	4.5	4.5	4.5	9	4.5	10
Borda	1	2	3	4	5	6	7	8	9	10
Dowdall	1	3	2	4	5	7	9	6	8	10
Bucklin	1	3.5	2	7.5	3.5	7.5	7.5	7.5	7.5	7.5
Instant Runoff Voting	1	3	2	8	4	9	10	6	5	7
Coombs	1	2	3	5	4	6	7	9	8	10
Nanson	1	2.5	2.5	4	7.5	7.5	7.5	7.5	7.5	7.5
Baldwin	1	2	3	4	6	7	8	5	9	10
Dodgson	1	2.5	2.5	4	5	6	8	7	9	10
Copeland	1	2	3	4	5.5	5.5	7	8.5	8.5	10
Black	1	2	3	4	5	6	7	9	8	10
Simpson–Kramer	1	3.5	2	3.5	8.5	6	6	8.5	6	10
Ranked pairs	1	2	3	4	5	6	7	9	8	10
Smith	1	2	3	4	7	7	7	7	7	10
Schwartz	1	2	3	4	7	7	7	7	7	10
Schulze	1	2	3	4	6.5	5	6.5	9	8	10
Quandt	1	2	3	4	5	6	7.5	7.5	9	10
Majority judgment	1	2	3	5	4	7	8	6	9	10
Borda Majority Count	1	2	3	4	5	6	7	8	9	10
Mean-Median Compromise Method *	1	2	3	5	4	6	8	7	9	10

*: Ranking obtained after tie-breaking.

When it comes to the official ranking of red wines—the one determined by range voting—MMCM and the official result of the 1976 Paris wine taste coincide exactly. Although opinions on the first two wines' rankings are divided, there is agreement on the rankings of the next eight red wines. By flipping the ranks of the fourth and fifth on one side and the seventh and eighth on the other, MMCM ranks white wines differently than the official result. The other wines are in the same range. The majority judgment upholds the MMCM's ranking with the exception of the wines ranked sixth and seventh.

8.2. The 2012 French Presidential Poll by OpinionWay

Using the dataset from the Terra Nova and OpinionWay presidential poll, we evaluate the efficacy of the MMCM. The voting data from 993 participants in the 2012 French presidential election, as detailed in Table 6, serves as the basis for our comparative analysis between the traditional first-past-the-post (FPTP) system, the majority runoff (MR), the MJ, and the RV among all ten candidates [6]. This experiment seeks to elucidate the potential advantages and limitations of the MMCM in accurately reflecting voter preferences and enhancing the robustness of electoral outcomes.

8.2.1. Preferences of Voters

In this study, we use data from the OpinionWay poll shown in Table 6 to examine voter preferences from the 2012 French presidential election. This dataset provides a comprehensive snapshot of the electorate's choices and serves as a robust foundation for analyzing the effectiveness of various voting schemes. We compare the MMCM to previous systems in order to assess how effectively it represents voters' complex sentiments by looking at the distribution of preferences.

8.2.2. Applying MMCM to the 2012 French Presidential Election

In our analysis of the 2012 French presidential election, we will refer to the candidates using the following coding system for clarity and brevity:

- "H" for F. Hollande;

- "B" for F. Bayrou;
- "S" for N. Sarkozy;
- "M" for J.-L. Mélenchon;
- "DA" for N. Dupont-Aignan;
- "J" for E. Joly;
- "P" for P. Poutou;
- "LP" for M. Le Pen;
- "A" for N. Arthaud;
- "C" for J. Cheminade.

Hereafter, we will use this coding to enhance readability and focus on electoral schemes, simplifying discussions on voter preferences and facilitating comparisons among candidates.

Table 13 presents the MMCM scores for candidates in the 2012 French presidential poll, highlighting the evaluation results based on different values of k. Cells are shaded in various colors to denote instances of ties in the scores. To resolve these ties, the MMCM was reapplied using $k = 3$. The final ranking of the candidates is presented in the last row, indicating that candidate H ranks first, candidate M ranks fourth, candidate J ranks eighth, and candidate C ranks last.

Table 13. MMCM scores for candidates in the 2012 French presidential poll.

Candidates	H	B	S	M	DA	J	P	LP	A	C
$k = 2$	4	4	3	3	2	1.67	2	2.33	1.33	1.33
$k = 3$	3.86	3.71	3.29	3.14	2.14	2	2	–	1.71	1.57
Ranking	1	2	3	4	6	8	7	5	9	10

8.2.3. Results of Other Voting Schemes

Table 14 reveals various insights about candidate rankings across four voting schemes: MMCM, MJ, RV, and FPTP. At a glance, it is evident that François Hollande (H) consistently ranks first in all voting schemes, highlighting his strong support among voters. Conversely, Nathalie Arthaud (A) and Jacques Cheminade (C) both occupy the lower end of the rankings, with Cheminade ranked last across all schemes.

A closer examination reveals some intriguing nuances in the rankings. For instance, while Bayrou (B) maintains a stable second place across MMCM, MJ, and RV, he drops to fifth in the FPTP system. This disparity implies that, as demonstrated by the MMCM results, voters' broader preferences for a more consensus-oriented approach are better captured by MMCM, MJ, and RV than by the FPTP process.

Table 14. Candidate rankings under different voting schemes.

Voting Scheme	MMCM	MJ	RV	FPTP
H	1	1	1	1
B	2	2	2	5
S	3	3	3	2
M	4	4	4	4
DA	6	5	6	7
J	8	6	7	6
P	7	7	8	8
LP	5	8	5	3
A	9	9	9	9
C	10	10	10	10

The results indicate that the Mean-Median Compromise Method (MMCM) provides a balanced representation of voter preferences, returning scores that range between the median and mean. This characteristic positions MMCM as a compromise between the extremes of the MJ and RV methods, suggesting it may effectively capture a broader consensus among voters. The consistency and fairness of MMCM are evident in candidate B's stable ranking, contrasting with the variability observed in FPTP. Additionally, MMCM effectively addresses ties and provides a more accurate reflection of voter preferences, underscoring its potential to improve electoral outcomes over traditional methods.

9. Discussion and Conjectures on MMCM

In this section, we engage in a thoughtful examination of MMCM's performance, offering conjectures and insights into its behavior across diverse electoral conditions. By evaluating robustness, comparing with single-peaked distributions, and investigating convergence to social consensus, we offer a positive outlook on the potential and distinctive benefits of MMCM in diverse voting scenarios.

9.1. MMCM vs. Single-Peakedness

A single-peaked distribution occurs when voter preferences P over a set of candidates $\{c_1, c_2, \ldots, c_m\}$ follow a unimodal pattern. Mathematically, for a preference profile to be single-peaked, there must exist a linear order $<$ on the candidates such that for every voter v_i, there is a candidate c (the peak) where preferences decrease monotonically as one moves away from c. That is, if $c_x < c < c_y$, then $P(c_x) \leq P(c)$, and $P(c_y) \leq P(c)$. In simple terms, a single-peaked distribution occurs when voters' preferences rise to a maximum grade for a single option and symmetrically decrease as they grade alternatives further from this peak, creating a single peak with no preference reversals.

When dealing with a single-peaked distribution of voter preferences, the MMCM may exhibit the following behavior:

- (B1) Balanced outcome: MMCM aims to find a compromise between different preferences by considering both the mean and the median of the assigned grades. In a single-peaked distribution, where voter preferences are concentrated around a central peak, MMCM is likely to produce a balanced outcome that reflects the central tendency of the distribution.
- (B2) Moderation of extremes: MMCM's emphasis on compromise may lead to the moderation of extreme preferences. Due to its robustness, voters with extreme preferences on either end of the distribution may find their influence mitigated, as MMCM seeks to find a consensus position that accommodates a broad range of preferences. Computer simulations conducted on 10^8 voters, with the number of divisions k taking values up to 14 (see Figure 2), show that even when 50% of voters replace their true evaluations with extreme ones, the variation in the MMCM score does not exceed 12%.

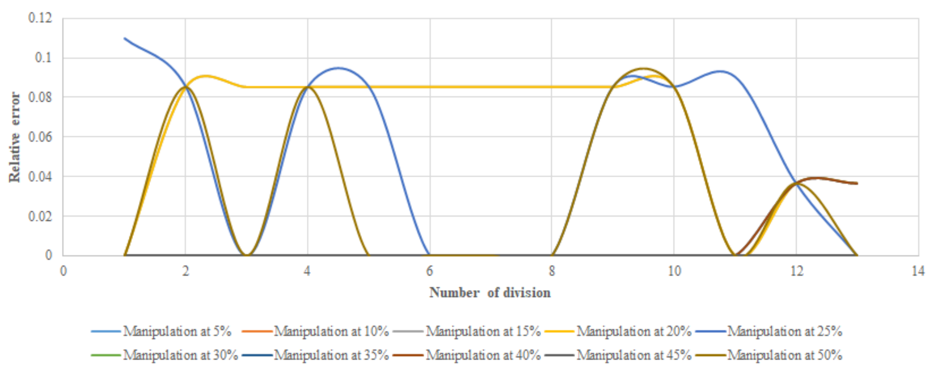

Figure 2. Moderation of extremes with the MMCM.

- (B3) Consistent result: Due to its systematic approach to combining mean and median values, MMCM is likely to produce consistent results that align with the central tendency of the distribution. This consistency helps ensure that the outcome is robust and reflective of the overall shape of the single-peaked distribution.

- (B4) Influence of outliers: While MMCM aims to balance different preferences, outliers or extreme grades may still have some influence on the final outcome, particularly if they significantly impact the mean or median values. However, the overall effect of outliers is likely to be moderated by the presence of a single peak in the distribution.

9.2. Strategy-Proofness

Take a look at Table 6. The intermediate grades of F. Hollande for $k = 2$ are 2, 4, and 6. As a result, F. Hollande's final MMCM score is 4. Assume that F. Hollande has been assigned a grade of g_j by voter v_j. The following circumstances are possible:

- If $g_j \geq 6$, then v_j can no longer raise F. Hollande's final score. He can, however, decrease it by assigning a grade that is less than 6. Given that F. Hollande's final score is already significantly lower than v_j had anticipated, it would not be prudent for them to attempt to lower it even further.
- When $4 \leq g_j \leq 6$, v_j has the ability to either raise or decrease F. Hollande's final score, depending on whether he receives a score greater than 6 or lower than 4.
- If $2 \leq g_j \leq 4$, then v_j can raise F. Hollande's final score if he or she assigns him a new score higher than 4. Giving him a score of less than 2 is another way for him or her to obtain a lower final score for F. Hollande.
- If $g_j \leq 2$, it is impossible for v_j to lower F. Hollande's score. Although it is not reasonably their intention—F. Hollande's score is already far better than he or she wanted it to be—they can still increase it by awarding him a score greater than 2.

These examples demonstrate that MMCM is a useful statistical parameter that deters strategic voting even if it is not entirely robust. Its lower robustness than MJ is, in our opinion, advantageous because a statistical measure that is too robust runs the risk of hiding important differences in the data. This may result in incorrect conclusions or an underestimating of the risks, particularly in industries like banking or medicine where exact variations are essential for making decisions. Readers who are interested might review the debates on robust statistical parameters and implications in [36].

9.3. Robustness and Sensitivity

In the context of grading, robustness of a social choice rule can be mathematically defined as the property that ensures the outcome remains stable and consistent when voter grades are slightly perturbed. Formally, if $f(G)$ is the social choice function applied to the set of grades G, then f is robust if, for any small perturbation ϵ of the grades, the outcome $f(G + \epsilon) \approx f(G)$. This property ensures that minor changes in individual grades do not significantly affect the overall social outcome.

To assess the robustness of MMCM under manipulative voting behavior, we conducted extensive simulations. Initially, we analyzed the distribution of grades assigned by voters to a candidate and computed the MMCM score for this candidate. Subsequently, we selected 5% of voters at random. If a voter's original grade was less than the candidate's MMCM score, we altered this grade to the minimum possible value. Conversely, if the grade was greater than or equal to the MMCM score, we changed it to the maximum possible value.

We repeated this simulation with 10^8 voters, varying the percentage of manipulated voters at increments of 5%, up to 50%. Our results, illustrated in Figure 2, demonstrate that even when up to 50% of voters replace their true evaluations with extreme grades, the variation in the MMCM score does not exceed 12%. This finding underscores MMCM's robustness in mitigating the influence of extreme voting behaviors and preserving the integrity of the consensus outcome.

9.4. Convergence to Social Consensus

We define the social consensus as the state that represents the most widely accepted outcome among voters. In this context, "most widely accepted" refers to the final score that a voting function returns for a candidate, reflecting the degree of satisfaction among the majority of voters. This outcome is derived from the aggregated grades assigned to the candidates by the voters. Thus, the score indicates the candidate's level of acceptance within the electorate. The candidate with the highest score embodies the consensus, as their score represents the broadest agreement across the voting population. Therefore, the social consensus is not merely a reflection of a simple majority but an amalgamation of all voter preferences, harmonized into a singular, representative score.

Achieving social consensus in voting is one of the fair properties that MMCM meets. Extreme grades can cause methods like range voting to fail, as demonstrated in Section 3. While majority judgment resists outliers, it has limitations that MMCM addresses better. MMCM effectively balances mean and median scores, reducing the impact of extreme grades and integrating a wider range of voter inputs for a more representative outcome.

10. Concluding Remarks

The purpose of this article is to persuade readers of several key facts about MMCM. The MJ developed by Balinski and Laraki has an unsuitable tie-breaking mechanism and remains overly robust. This extreme robustness can be perceived as a disadvantage rather than a benefit. The following points have been discussed and/or proven:

- The majority judgment is a L^p deepest voting with $p = 1$.
- Range voting is a L^p deepest voting with $p = 2$.
- MJ's tie-breaking system is incompatible with the deepest voting theory.
- MMCM is a deepest voting procedure of which MJ (without its tie-breaking mechanism) is a special case.
- The different MMCM tie-breaking procedures include L^p deepest voting, with p ranging from 1 to 2.
- As a method of grading, MMCM fulfills the conditions of universality, anonymity, neutrality, monotony, and independence of irrelevant alternatives.
- To avoid the tyranny of the majority, MMCM is meant to be less robust than MJ (with a division number of $k = 2$ at the beginning).
- Due to its membership in the L^p depths family, MMCM is vulnerable to no-show, reinforcement, and Condorcet paradoxes, among others.
- The only voting process in the L^p depths family that is pivotal, regular, and allows the arithmetic mean to be a characterizer is MMCM.
- When MMCM is used for real-world elections, such as the 1976 Paris wine taste and the 2012 French presidential poll, consistent outcomes are obtained.

Based on the foregoing, we conclude that MMCM, whose tie-break mechanisms are L^p depths with p varying between 1 and 2, is the best compromise between MJ ($p = 1$) and RV ($p = 2$).

The Mean-Median Compromise Method has shown its effectiveness in real-world scenarios like the 1976 Paris wine contest and the 2012 French presidential poll by OpinionWay. To better understand its robustness, it needs further testing on diverse datasets. Machine learning can enable dynamic adjustments in the degree of division k, enhancing its adaptability to preferences. Ethical and cultural considerations could help ensure respect for population variations. By focusing on adaptive modeling, empirical validation, and ethical issues, we can increase the method's applicability and impact.

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Abbreviations

The following abbreviations are used in this manuscript:

AV	Approval Voting
FPTP	First-Past-The-Post
IIA	Independence of Irrelevant Alternatives
MJ	Majority Judgment
MMCM	Mean-Median Compromise Method
MR	Majority Runoff
RV	Range Voting

Appendix A

- Plurality: Every voter fills out a ballot with the name of one option listed. The winner is the candidate with the most votes.
- Negative Plurality: Every voter submits a ballot with the candidates listed in complete linear order. The candidates who receive the lowest number of votes in the final position are returned by this rule.
- Borda: Every voter submits a ballot with the candidates listed in complete linear order. For every vote that places a candidate in k^{th} place, that candidate receives a score of $m - k$. The winner is the contender with the highest overall Borda score. See [34,35] for more details.
- Dowdall: This method is an alternative to the Borda count. Voters must assign unique preference scores to each candidate, with no two candidates sharing the same score. A voter's first choice receives a score of 1, the second choice receives 1/2, the third choice receives 1/3, and so on. The candidate with the highest total score wins.
- Bucklin: Voters may cast ballots based on rank preference. Votes for top choices are tallied first. A candidate wins if they have a majority of the vote. If not, the initial selections are supplemented with the second choices. Once more, the candidate with the most total votes is the winner if a contender receives a majority of the vote. As needed, lower rankings are added. The person who requires the fewest extra votes to receive a majority wins if there is a tie in the stage.
- Instant Runoff Voting: According to their preferences, voters rate the candidates. First, ballots are tallied according to each voter's preference. Until one of the surviving candidates obtains a majority, the candidate who receives the fewest votes at the beginning of each round is eliminated. See [35] for more details.
- Coombs: Until there is only one candidate with a strict majority, we keep eliminating the candidate who has received the most vetoes.
- Nanson: Every voter provides a rating of the contenders. They calculate the Borda score. Those candidates whose Borda score is the same as the average or lower are disqualified. After that, the procedure is repeated while computing a fresh Borda count on the lowered profile. This process complies with Condorcet. See [34] for more details.
- Condorcet: Every voter submits a ballot listing all of the candidates in complete linear order. In pairwise comparisons, the winner is the candidate who receives the majority of the votes over all other candidates. See [35] for more details.
- Baldwin: Nanson's voting rule and this rule are closely connected. Up until one candidate is left, it successively removes the candidate with the lowest Borda score. In the event that multiple candidates have the same score, one of them is eliminated by a tie-breaking method. See [34,35] for more details.

- Dodgson: If an alternative can be changed to make a Condorcet winner by swapping out as few adjacent alternatives in the individual rankings as possible, it is referred to as a Dodgson winner. See [35] for more details.
- Copeland: Every voter submits a ballot with the candidates listed in complete linear order. There are pairwise comparisons between the candidates. A candidate's Copeland score is calculated by deducting the number of opponents who defeat them from the total number of opponents they defeat. See [35] for more details.
- Black: There are two phases to this rule. First, we ascertain whether a Condorcet winner exists. This is the victor, if there is one. If not, we return the Borda rule's outcome.
- Simpson–Kramer: Every voter submits a ballot with the candidates listed in complete linear order. The winner is the candidate against whom the smallest majority (in favor of another candidate) can be gathered. See [35] for more details.
- Ranked pairs: If there exists a Condorcet winner, the rule selects her or him. Otherwise, just bypass the head-to-head outcome which will activate a loop. Ref. [37] points out that even after taking into account every victory, there might still be more than one possible ranking if there are ties. In this instance, the candidates who rank first in the remaining possible rankings are deemed to be tied. Tideman has dubbed this rule "ranked pairs" for this reason. See [30] for more details.
- Smith: This rule results in the election of every candidate in the Smith set, which is the smallest non-empty set of candidates such that, in pairwise elections, every candidate in the set defeats every candidate outside the set. The candidate is unique in the Smith set when there is a Condorcet winner. See [35] for more details.
- Schwartz: Every candidate in the Schwartz set is elected by this rule. The union of all the undominated sets forms the Schwartz set, which is a subset of the Smith set. If all candidates within a set are pairwise undefeated by all candidates outside the set, the set is said to be undominated; no non-empty proper subset can meet this requirement. When there is a Condorcet winner, the Schwartz set is a singleton. See [35] for more details.
- Schulze: From the one they most want to win to the one they least want to win, voters rate the candidates. More than one candidate may receive the same preference, and candidates may receive no number at all (which is seen to indicate that they are the worst). Next, for every pair of candidates, the two are compared: the number of voters who agree with the majority decision is noted. If t voters prefer x_1 to x_2, t voters prefer x_2 to x_3, and so on, and t voters prefer $x_{(m-1)}$ to x_m, then x_1 is said to beat x_m with strength t. The winner is the candidate who beats any other candidate with the largest strength t. See [38] for more details.
- Range voting: Voters are asked to provide a rating to each candidate; the total score of the candidates is calculated by adding (or averaging) the grades. The candidate who scores the highest wins the election. See [39–41] for more details.
- Borda Majority Count (BMC): Similar to range voting, this system also includes a tie-breaking mechanism in the event that the scores are equal. See [18,19] for more details.
- Majority Judgment: Voters are asked to allot grades to candidates in a well-known language. Median grades are computed for all candidates. The candidate with the highest median wins. Since median values are often equal, the method is provided with a tie-breaking mechanism. See [3] for more details.
- Quandt's method: As multiple candidates may receive the same grade from the judge, each is given the average of where they could fall in the judge's ranking (for example, the four candidates with the second-highest grade, who are ranked 3.5 each, are occupying spots 2, 3, 4, and 5 in the ranking). According to this technique, the candidates are ranked based on their sums, or equivalently, their averages, and the judges' individual rankings are treated as points. The contender with the lowest total number of points is declared the winner. See [33] for more details.

- Shapley ranking: An alternative to approval voting is this rule. Ref. [30] suggests that each judge's one vote unit be distributed evenly among the individuals in the group of their choice. These shares are tallied, candidate by candidate, just like in approval voting, and a final ranking is determined. It turns out that in an analogous cooperative game, each candidate's Shapley value is equal to the total "number of votes" connected with them [42]. The candidate who receives the most votes is ranked first among the others.

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